Computer Graphics

Lecture 03 – 2D and 3D Transformations

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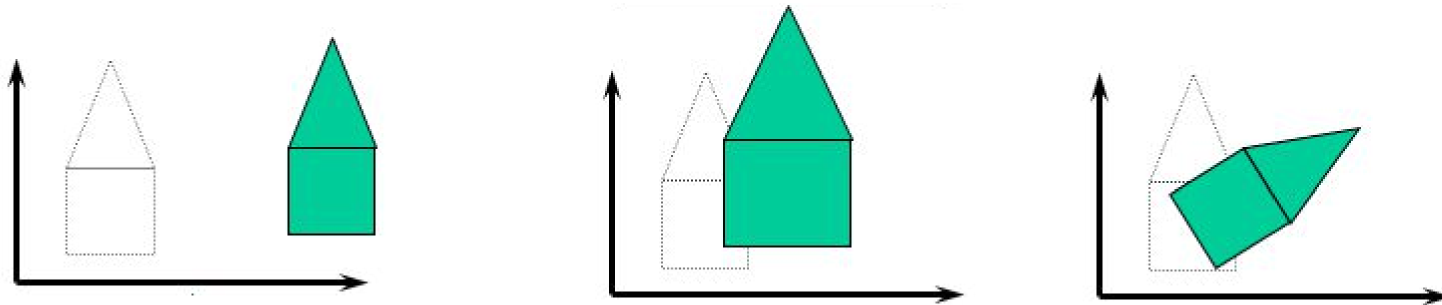
2D Transformations

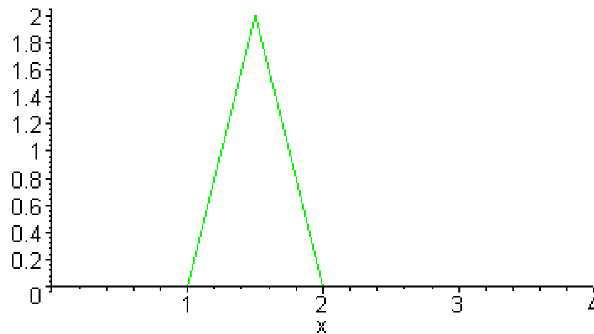
- Transformations are a fundamental part of computer graphics. They can be used to **position objects**, **shape objects**, **change viewing positions**, and even to change how something is viewed (projection transformation).

- Let's start with 2D transformations: translation, scaling and rotation.



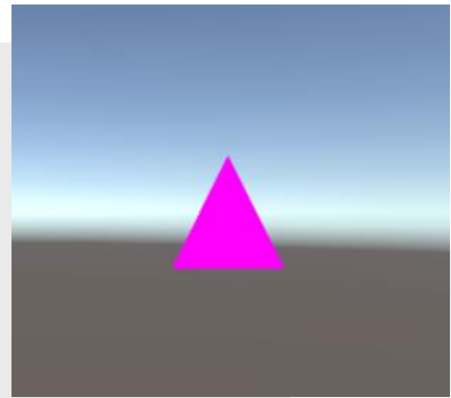
Representation of Points/Objects

- 2D objects are represented by a set of points (vertices), $\{p_1, p_2, ..., p_n\}$, and an associated set of edges $\{e_1, e_2, ..., e_m\}$, where each edge is a pair of points $e = \{p_i, p_j\}$.



$$p = \begin{bmatrix} x \\ y \end{bmatrix}$$

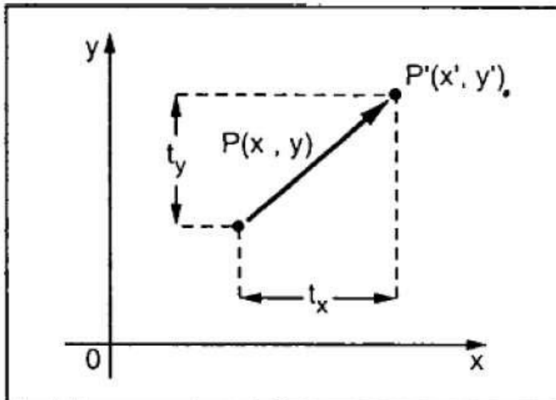
- We can use Unity to create a mesh with a triangle (ignoring the z coordinate).



```
...

private Mesh mesh;
private Vector3[] vertices;

void Start(){
  mesh = new Mesh();
  GetComponent<MeshFilter>().mesh = mesh;
  mesh.name = "MyMesh";
  vertices = new Vector3[3];
  vertices[0] = new Vector3(3, 0, 1);
  vertices[1] = new Vector3(5, 0, 1);
  vertices[2] = new Vector3(4, 2, 1);
  mesh.vertices = vertices;
  int[] triangles = new int[3];
  triangles[0] = 0;
  triangles[1] = 2;
  triangles[2] = 1;
  mesh.triangles = triangles;
}

...
```

2D Translation

- A translation moves an object to a different position on the screen. A point can be translated by adding a translation coordinate (t_x, t_y) to the original coordinate (X, Y) to get the new coordinate (X', Y').



$$X' = X + t_x$$

$$Y' = Y + t_y$$

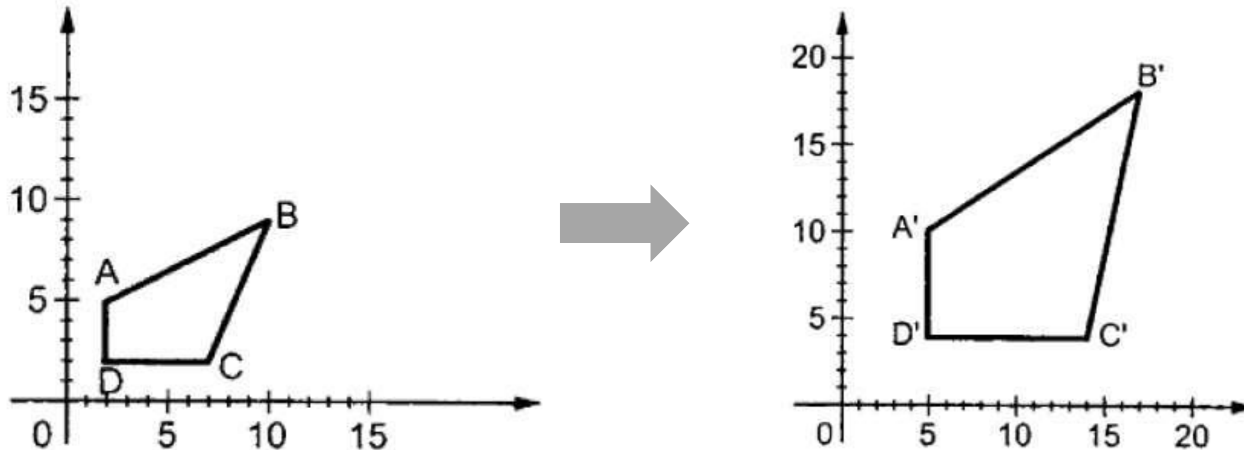
2D Translation

```
void Translate2D(float tx, float ty)
{
  for (int i = 0; i < vertices.Length; i++)
  {
    vertices[i] = new Vector3(vertices[i].x + tx,
                              vertices[i].y + ty);
  }
  mesh.vertices = vertices;
}
```

```
void Update()
{
  Translate2D(2 * Time.deltaTime, 0);
}
```

2D Scaling

- Scaling can be achieved by multiplying the original coordinates of the object with the scaling factor (S_x, S_y) . The object can be either expanded compressed.



$$X' = X \times S_x$$

$$Y' = Y \times S_y$$

2D Scaling

```
void Scale2D(float tx, float ty)
{
  for (int i = 0; i < vertices.Length; i++)
  {
    vertices[i] = new Vector3(vertices[i].x * sx,
                              vertices[i].y * sy);
  }
  mesh.vertices = vertices;
}
```

```
void Update()
{
  Scale2D(1.01f, 1);
}
```

2D Scaling – Matrix

- We can also represent the scaling transformation using matrix multiplication:

$$X' = X \times S_x$$
$$Y' = Y \times S_y$$

$$\begin{bmatrix} X' \\ Y' \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} \times \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix}$$

- How it works?

1	2	×	a	c	=	1a + 2b	1c + 2d
3	4		b	d		3a + 4b	3c + 4d

$$\begin{bmatrix} X \\ Y \end{bmatrix} \times \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix} = \begin{bmatrix} X \times S_x + X \times 0 \\ Y \times 0 + Y \times S_y \end{bmatrix}$$

2D Scaling – Matrix

```
void Scale2DMat(float sx, float sy)
{
  float[,] mat = new float[2, 2];
  mat[0, 0] = sx; mat[0, 1] = 0;
  mat[1, 0] = 0;  mat[1, 1] = sy;
  for (int i = 0; i < vertices.Length; i++)
  {
    vertices[i] = multiply(mat, vertices[i]);
  }
  mesh.vertices = vertices;
}
```

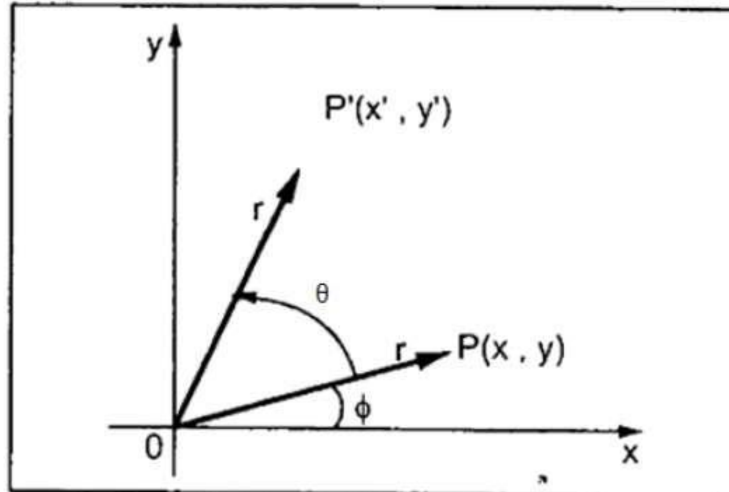
```
void Update()
{
  Scale2DMat(1.01f, 1);
}
```

Point × Matrix

```
Vector3 multiply(float[,] matrix, Vector3 point)
{
  Vector3 result = new Vector3();
  for (int r = 0; r < matrix.GetLength(0); r++)
  {
    float s = 0;
    for (int z = 0; z < matrix.GetLength(1); z++)
      s += matrix[r, z] * point[z];
    result[r] = s;
  }
  return result;
}
```

2D Rotation

- In rotation, we rotate the object at particular angle θ (theta) from its origin.



$$X' = X \times \cos(\theta) - Y \times sin(\theta)$$

$$Y' = X \times \sin(\theta) + Y \times cos(\theta)$$

2D Rotation

```
void Rotate2D(float tx, float ty)
{
  for (int i = 0; i < vertices.Length; i++)
  {
    vertices[i] = new Vector3(vertices[i].x * Mathf.Cos(angle)
                            - vertices[i].y * Mathf.Sin(angle),
                              vertices[i].x * Mathf.Sin(angle)
                            + vertices[i].y * Mathf.Cos(angle));
  }
  mesh.vertices = vertices;
}
```

```
void Update()
{
  Rotate2D(20 * Mathf.Deg2Rad * Time.deltaTime);
}
```

2D Rotation – Matrix

- We can also represent rotation transformations using matrix multiplication:

$$X' = X \times \cos(\theta) - Y \times sin(\theta)$$

$$Y' = X \times \sin(\theta) + Y \times cos(\theta)$$

$$\begin{bmatrix} X' \\ Y' \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} \times \begin{bmatrix} cos\,\theta & -\,sin\,\theta \\ sin\,\theta & cos\,\theta \end{bmatrix}$$

2D Rotation – Matrix

```
void Rotate2DMat(float angle)
{
  float[,] mat = new float[2, 2];
  mat[0, 0] = Mathf.Cos(angle); mat[0, 1] = -Mathf.Sin(angle);
  mat[1, 0] = Mathf.Sin(angle); mat[1, 1] = Mathf.Cos(angle);
  for (int i = 0; i < vertices.Length; i++)
  {
    vertices[i] = multiply(mat, vertices[i]);
  }
  mesh.vertices = vertices;
}
```

```
void Update()
{
  Rotate2DMat(20 * Mathf.Deg2Rad * Time.deltaTime);
}
```

2D Translation – Matrix

- Can we represent translation using matrix multiplication?

$$X' = X + t_x$$

$$Y' = Y + t_y$$

  - We need a 2 by 2 matrix, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, such that $X \times a + A \times c = X + t_x$.
    - There is no way to obtain the t_x term!
  - We can't represent translation using a 2 by 2 matrix!

2D Homogeneous Coordinates

- What is required at this point is to change the setting (2D coordinate space) in which we phrased our original problem.
  - In geometry, a common strategy to solve a problem in *n* space consists in rephrasing it to a *n+1* space.

$$\begin{bmatrix} X \\ Y \end{bmatrix} \rightarrow \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} \rightarrow \text{ Transformation } \rightarrow \begin{bmatrix} X' \\ Y' \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} X' \\ Y' \end{bmatrix}$$

- In other words, the homogeneous coordinate system adds an extra virtual dimension. Thus 2D homogeneous coordinates are actually 3D. This kind of transformation is called an affine transformation.

2D Homogeneous Coordinates

- When using homogeneous coordinates to represent transformations it is important to distinguish vectors that represent positions and directions.

$\begin{bmatrix}10\\10\\1\end{bmatrix}$ is a position. $\begin{bmatrix}1\\1\\0\end{bmatrix}$ is a direction.

- This is important because vectors that represent <u>directions or offsets should not change</u> when we translate an object.
  - If there is a scaling/rotation transformation in the upper-left 2×2 entries of the matrix, it will apply to the vector, but the translation still multiplies with the zero and is ignored.

2D Translation – Homogeneous Coordinates

- Now we can represent translations as a matrix multiplication:

$$\begin{bmatrix} X' \\ Y' \\ 1 \end{bmatrix} = \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

```
void Translate2DMatHM(float tx, float ty){
  float[,] mat = new float[3, 3];
  mat[0, 0] = 1; mat[0, 1] = 0; mat[0, 2] = tx;
  mat[1, 0] = 0; mat[1, 1] = 1; mat[1, 2] = ty;
  mat[2, 0] = 0; mat[2, 1] = 0; mat[2, 2] = 1;
  for (int i = 0; i < vertices.Length; i++){
    vertices[i] = multiply(mat, vertices[i]);
  }
  mesh.vertices = vertices;
}
```

2D Scale – Homogeneous Coordinates

- We can also represent scale as a matrix multiplication in homogeneous coordinates:

$$\begin{bmatrix} X' \\ Y' \\ 1 \end{bmatrix} = \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} \times \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

```
void Scale2DMatHM(float sx, float sy){
  float[,] mat = new float[3, 3];
  mat[0, 0] = sx; mat[0, 1] = 0;  mat[0, 2] = 0;
  mat[1, 0] = 0;  mat[1, 1] = sy; mat[1, 2] = 0;
  mat[2, 0] = 0;  mat[2, 1] = 0;  mat[2, 2] = 1;
  for (int i = 0; i < vertices.Length; i++){
    vertices[i] = multiply(mat, vertices[i]);
  }
  mesh.vertices = vertices;
}
```

2D Rotation – Homogeneous Coordinates

- We can also represent rotation as a matrix multiplication in homogeneous coordinates:

$$\begin{bmatrix} X' \\ Y' \\ 1 \end{bmatrix} = \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} \times \begin{bmatrix} cos\,\theta & -\,sin\,\theta & 0 \\ sin\,\theta & cos\,\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

```
void Rotate2DMatHM(float angle){
  float[,] mat = new float[3, 3];
  mat[0,0]=Mathf.Cos(angle); mat[0,1]=-Mathf.Sin(angle); mat[0,2]=0;
  mat[1,0]=Mathf.Sin(angle); mat[1,1]=Mathf.Cos(angle);  mat[1,2]=0;
  mat[2,0]=0;                mat[2,1]=0;                 mat[2,2]=1;
  for (int i = 0; i < vertices.Length; i++){
    vertices[i] = multiply(mat, vertices[i]);
  }
  mesh.vertices = vertices;
}
```

Combining Transformations

- Using homogeneous coordinates, every rotation, translation, and scaling operation can be represented by a **matrix multiplication**.

- The advantages of the homogeneous coordinates became clear when we need to combine more than one transformation at once.
  - Applying transformation matrices in sequence is the same as applying the product of those matrices once.

- This is a **key concept** that underlies most graphics hardware and software.

Combining Transformations – Example 1

- For example, suppose we wish to first rotate an object 90 degrees and then scale the object by 2 along the x axis.

$$\text{rotate}(90) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{scale}(2,1) = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{rotate}(90) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{scale}(2,1) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  – When combining transformations, it is very important to remember that matrix multiplication is not commutative. So the order of transforms does matter. The transforms are applied from the *right side first*.

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Combining Transformations – Example 1

$$rotate(90) + scale(2,1) = \begin{bmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

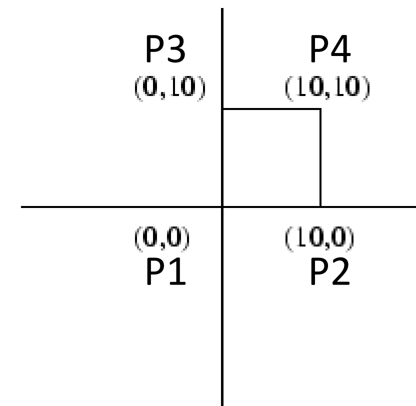
$$P1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$P2 = \begin{bmatrix} 10 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 1 \end{bmatrix}$$

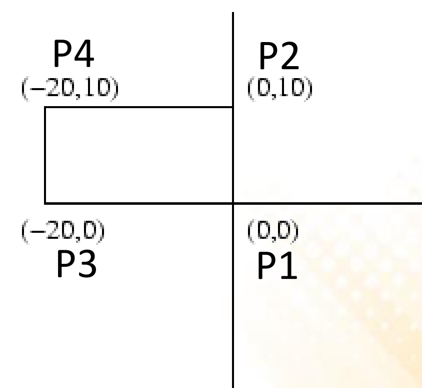
$$P3 = \begin{bmatrix} 0 \\ 10 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -20 \\ 0 \\ 1 \end{bmatrix}$$

$$P4 = \begin{bmatrix} 10 \\ 10 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -20 \\ 10 \\ 1 \end{bmatrix}$$

Square object:



Rotated and scaled square:



Combining Transformations – Example 1

```
float[,] multiply(float[,] matrix1, float[,] matrix2)
{
  float[,] result = new float[matrix1.GetLength(0),
                              matrix2.GetLength(1)];
  for (int i = 0; i < matrix1.GetLength(0); i++)
  {
    for (int j = 0; j < matrix2.GetLength(1); j++)
    {
      float s = 0;
      for (int k = 0; k < matrix1.GetLength(1); k++)
      {
        s += matrix1[i, k] * matrix2[k, j];
      }
      result[i, j] = s;
    }
  }
  return result;
}
```

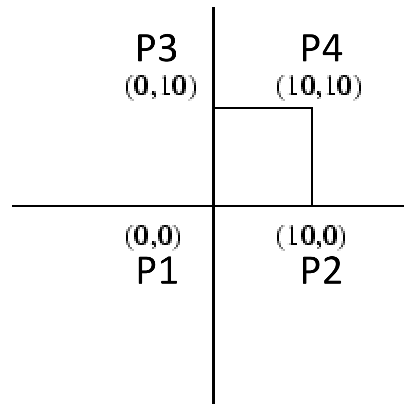
Combining Transformations – Example 1

```
void RotateScale(float angle, float sx, float sy)
{
  float[,] rmat = new float[3, 3];
  rmat[0,0]=Mathf.Cos(angle);rmat[0,1]=-Mathf.Sin(angle);rmat[0,2]=0;
  rmat[1,0]=Mathf.Sin(angle);rmat[1,1]=Mathf.Cos(angle); rmat[1,2]=0;
  rmat[2,0]=0;               rmat[2,1]=0;                 rmat[2,2]=1;

  float[,] smat = new float[3, 3];
  smat[0,0] = sx; smat[0,1] = 0;  smat[0,2] = 0;
  smat[1,0] = 0;  smat[1,1] = sy; smat[1,2] = 0;
  smat[2,0] = 0;  smat[2,1] = 0;  smat[2,2] = 1;

  float[,] finalmat = multiply(scalemat, rmat);
  for (int i = 0; i < vertices.Length; i++)
  {
    vertices[i] = multiply(finalmat, vertices[i]);
  }
  mesh.vertices = vertices;
}
```

Combining Transformations – Example 2

- In order to rotate an object around a specific point (such as its center or a corner). We need to perform 3 operations:
  - (1) translate the object to the origin;
  - (2) rotate the object;
  - (3) translate the object back to its initial position.

- Suppose we wish to rotate our object 90 degrees around its upper right corner (P4).



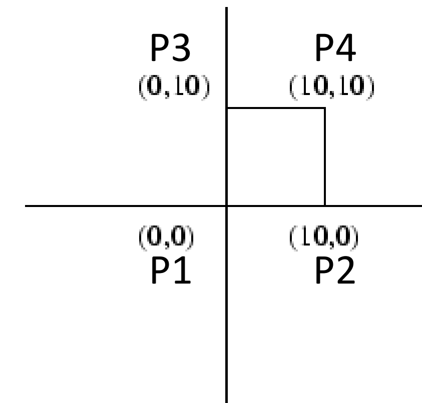
Combining Transformations – Example 2

- Rotation of 90 degrees around the upper right corner (P4 = (10, 10)):

$$T_1 = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -10 \\ 0 & 1 & -10 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} cos\,\theta & -sin\,\theta & 0 \\ sin\,\theta & cos\,\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_2 = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & 10 \\ 0 & 0 & 1 \end{bmatrix}$$



$$T_1 \times R \times T_2 = \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & 10 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & -10 \\ 0 & 1 & -10 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 20 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Combining Transformations – Example 2

- Rotation of 90 degrees around the upper right corner (P4 = (10, 10)):

$$T_1 \times R \times T_2 = \begin{bmatrix} 0 & -1 & 20 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

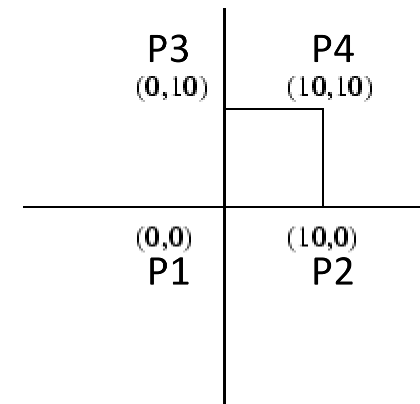
$$P1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 & -1 & 20 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \\ 1 \end{bmatrix}$$

$$P2 = \begin{bmatrix} 10 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 & -1 & 20 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 20 \\ 10 \\ 1 \end{bmatrix}$$

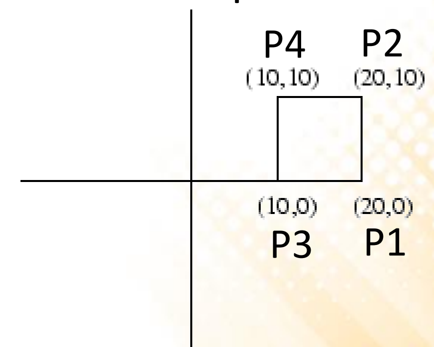
$$P3 = \begin{bmatrix} 0 \\ 10 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 & -1 & 20 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 1 \end{bmatrix}$$

$$P4 = \begin{bmatrix} 10 \\ 10 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 & -1 & 20 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 1 \end{bmatrix}$$

Square object:



Rotated square:



Combining Transformations – Example 2

```
void RotateAroundPoint(float angle, float px, float py)
{
  float[,] rotmat = new float[3, 3];
  rmat[0,0]=Mathf.Cos(angle);rmat[0,1]=-Mathf.Sin(angle);rmat[0,2]=0;
  rmat[1,0]=Mathf.Sin(angle);rmat[1,1]=Mathf.Cos(angle); rmat[1,2]=0;
  rmat[2,0]=0;               rmat[2,1]=0;                 rmat[2,2]=1;
  float[,] tran1mat = new float[3, 3];
  t1mat[0, 0] = 1; t1mat[0, 1] = 0; t1mat[0, 2] = -px;
  t1mat[1, 0] = 0; t1mat[1, 1] = 1; t1mat[1, 2] = -py;
  t1mat[2, 0] = 0; t1mat[2, 1] = 0; t1mat[2, 2] = 1;
  float[,] tran2mat = new float[3, 3];
  t2mat[0, 0] = 1; t2mat[0, 1] = 0; t2mat[0, 2] = px;
  t2mat[1, 0] = 0; t2mat[1, 1] = 1; t2mat[1, 2] = py;
  t2mat[2, 0] = 0; t2mat[2, 1] = 0; t2mat[2, 2] = 1;
  float[,] finalmat = multiply(multiply(t2mat, rmat), t1mat);
  for (int i = 0; i < vertices.Length; i++)
  {
    vertices[i] = multiply(finalmat, vertices[i]);
  }
  mesh.vertices = vertices;
}
```

Exercise 1

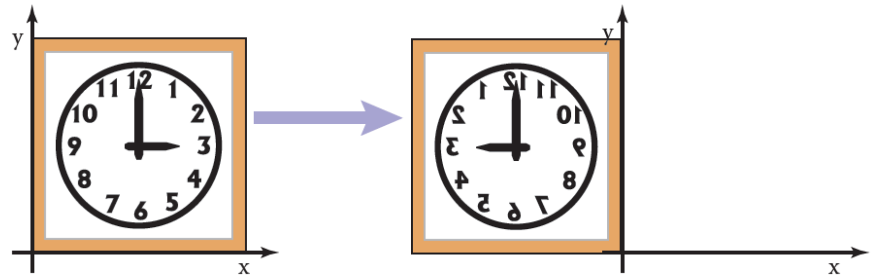
1) Reflection is a transformation that flips the object over the axis of reflection (x or y).

a) What is the reflection matrix for the x-axis?

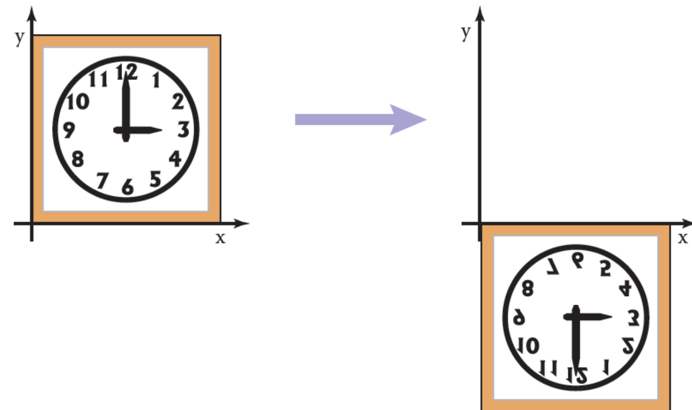
b) What is the reflection matrix for the y-axis?

c) Implement and apply the reflection transformation.

Reflection over the x-axis:



Reflection over the y-axis:



Exercise 2

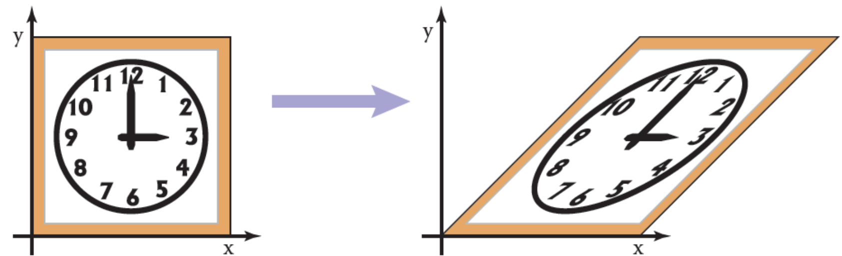
2) Shearing is a transformation that displaces each point in a fixed direction, by an amount proportional to its signed distance from a line that is parallel to that direction.

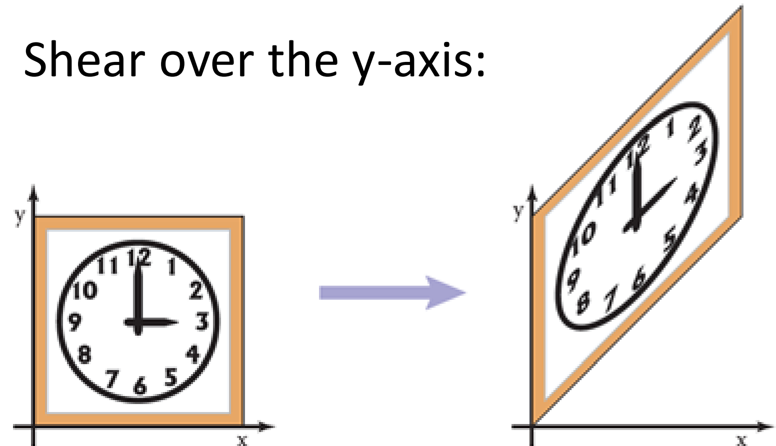
a) What is the shearing matrix for the x-axis?

b) What is the shearing matrix for the y-axis?

c) Implement and apply the shearing transformation.

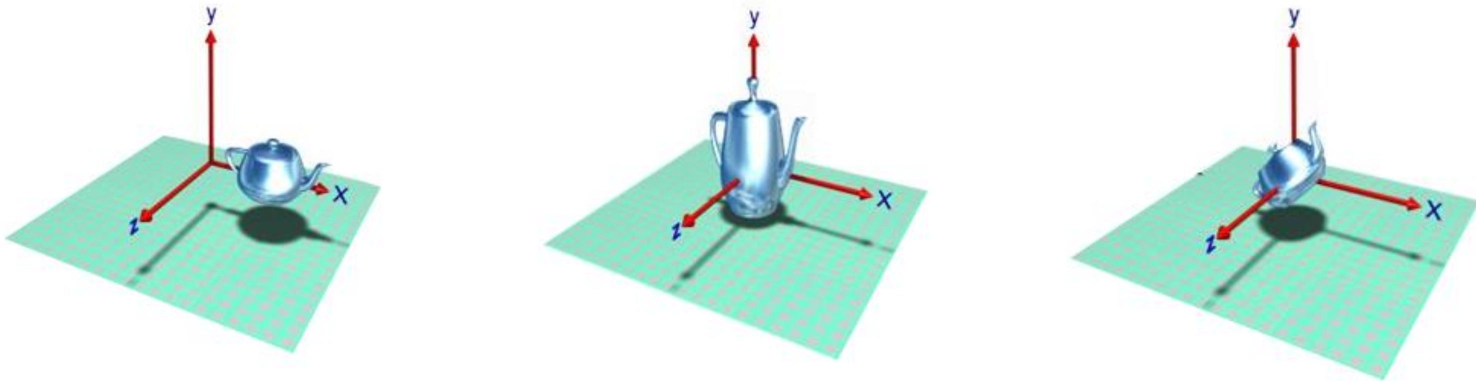
Shear over the x-axis:





3D Transformations

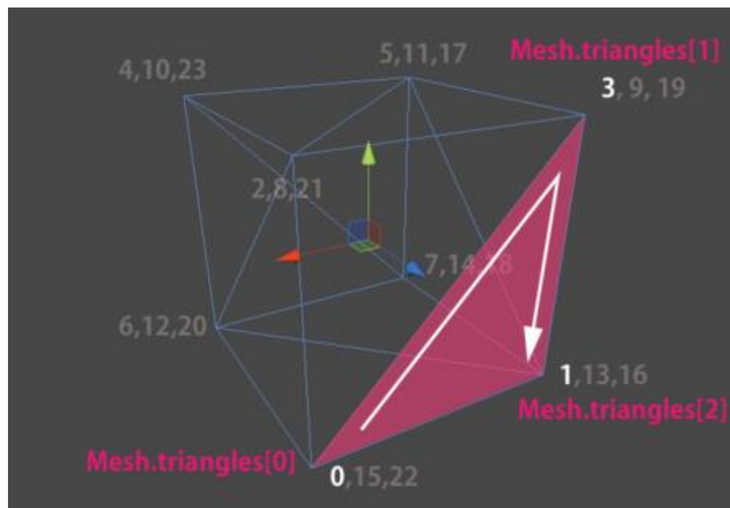
- 3D Transformations are an extension of the 2D transformations.

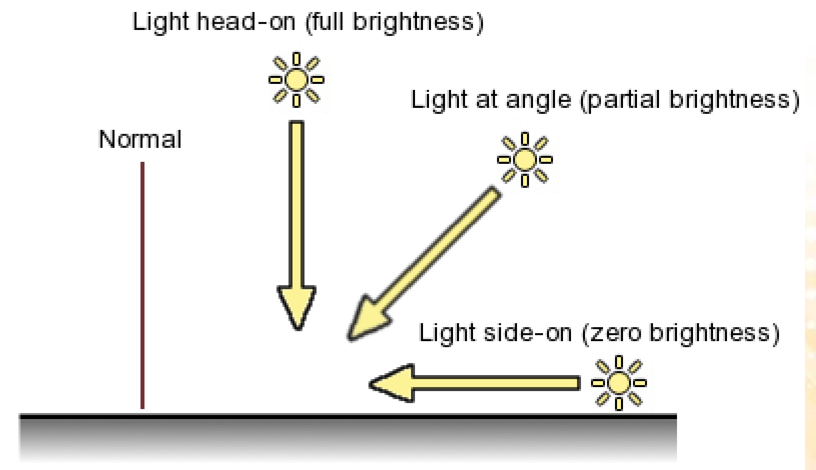


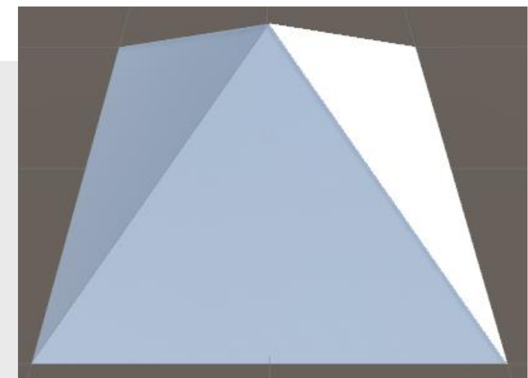
- We can also use **homogeneous coordinates in 3D** by adding a fourth coordinate to the transformation matrices.

Representation of 3D Objects

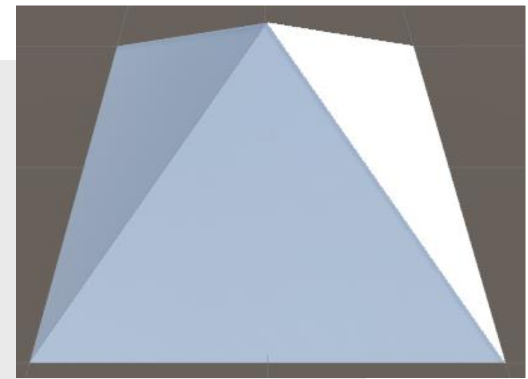
- 3D objects are represented by a **set of triangles**. Each triangle is defined by **three vertices**. In addition, each vertex has a **normal vector**, which usually points outward (perpendicular to the mesh surface).
  - The vertices and triangles define the spatial structure of the object.
  - The normals define how light affects the surface of the object.



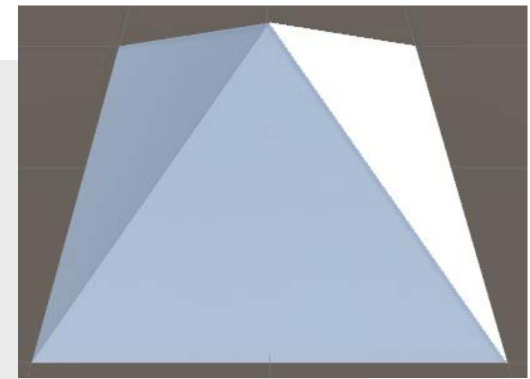




```
vertices = new Vector3[18];
//back triangle vertices
vertices[0] = new Vector3(-10, 0, -10);
vertices[1] = new Vector3(10, 0, -10);
vertices[2] = new Vector3(0, 10, 0);
//front triangle vertices
vertices[3] = new Vector3(-10, 0, 10);
vertices[4] = new Vector3(10, 0, 10);
vertices[5] = new Vector3(0, 10, 0);
//left triangle vertices
vertices[6] = new Vector3(-10, 0, 10);
vertices[7] = new Vector3(-10, 0, -10);
vertices[8] = new Vector3(0, 10, 0);
//right triangle vertices
vertices[9] = new Vector3(10, 0, 10);
vertices[10] = new Vector3(10, 0, -10);
vertices[11] = new Vector3(0, 10, 0);
//botton triangle 1 vertices
vertices[12] = new Vector3(-10, 0, -10);
vertices[13] = new Vector3(10, 0, -10);
vertices[14] = new Vector3(-10, 0, 10);
//botton triangle 2 vertices
vertices[15] = new Vector3(10, 0, 10);
vertices[16] = new Vector3(10, 0, -10);
vertices[17] = new Vector3(-10, 0, 10);
```



```
int[] triangles = new int[18];
//back triangle
triangles[0] = 0;
triangles[1] = 2;
triangles[2] = 1;
//front triangle
triangles[3] = 4;
triangles[4] = 5;
triangles[5] = 3;
//left triangle
triangles[6] = 6;
triangles[7] = 8;
triangles[8] = 7;
//right triangle
triangles[9] = 10;
triangles[10] = 11;
triangles[11] = 9;
//botton triangle 1
triangles[12] = 13;
triangles[13] = 14;
triangles[14] = 12;
//botton triangle 2
triangles[15] = 15;
triangles[16] = 17;
triangles[17] = 16;
```



```
Vector3[] normals = new Vector3[18];

normals[0] = Vector3.back + Vector3.up;
normals[1] = Vector3.back + Vector3.up;
normals[2] = Vector3.back + Vector3.up;

normals[3] = Vector3.forward + Vector3.up;
normals[4] = Vector3.forward + Vector3.up;
normals[5] = Vector3.forward + Vector3.up;

normals[6] = Vector3.left + Vector3.up;
normals[7] = Vector3.left + Vector3.up;
normals[8] = Vector3.left + Vector3.up;

normals[9] = Vector3.right + Vector3.up;
normals[10] = Vector3.right + Vector3.up;
normals[11] = Vector3.right + Vector3.up;

normals[12] = Vector3.down;
normals[13] = Vector3.down;
normals[14] = Vector3.down;

normals[15] = Vector3.down;
normals[16] = Vector3.down;
normals[17] = Vector3.down;
```

3D Translation – Homogeneous Coordinates

- Using homogeneous coordinates, we can also represent 3D translation as a matrix multiplication:

$$\begin{bmatrix} X' \\ Y' \\ Z' \\ 1 \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

```
void Translate3D(float tx, float ty, float tz){
  Matrix4x4 translation_matrix = new Matrix4x4();
  translation_matrix.SetRow(0, new Vector4(1f, 0f, 0f, tx));
  translation_matrix.SetRow(1, new Vector4(0f, 1f, 0f, ty));
  translation_matrix.SetRow(2, new Vector4(0f, 0f, 1f, tz));
  translation_matrix.SetRow(3, new Vector4(0f, 0f, 0f, 1f));
  for (int i = 0; i < vertices.Length; i++){
    vertices[i]=translation_matrix.MultiplyPoint(vertices[i]);
  }
  mesh.vertices = vertices;
}
```

3D Scale – Homogeneous Coordinates

- We can also represent 3D scale as a matrix multiplication:

$$\begin{bmatrix} X' \\ Y' \\ Z' \\ 1 \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \times \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

```
void Scale3D(float sx, float sy, float sz){
  Matrix4x4 scale_matrix = new Matrix4x4();
  scale_matrix.SetRow(0, new Vector4(sx, 0f, 0f, 0));
  scale_matrix.SetRow(1, new Vector4(0f, sy, 0f, 0));
  scale_matrix.SetRow(2, new Vector4(0f, 0f, sz, 0));
  scale_matrix.SetRow(3, new Vector4(0f, 0f, 0f, 1f));
  for (int i = 0; i < vertices.Length; i++){
    vertices[i]=scale_matrix.MultiplyPoint(vertices[i]);
  }
  mesh.vertices = vertices;
}
```

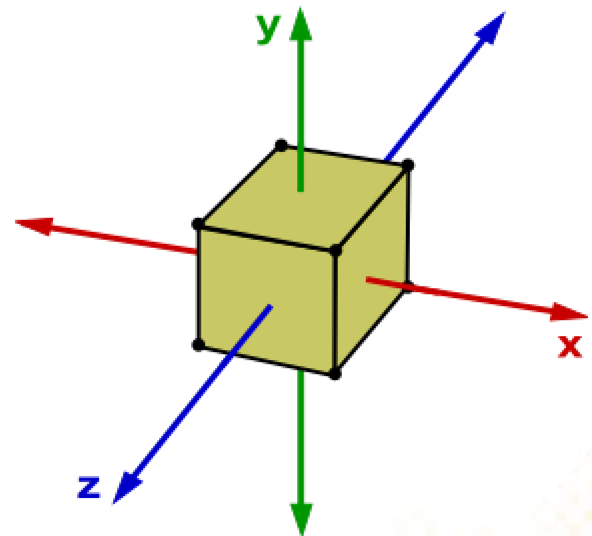
3D Rotation – Homogeneous Coordinates

- 3D rotations are not exactly same as 2D rotations. In 3D, we have to specify the angle of rotation along with the axis of rotation (we can perform 3D rotations about X, Y, and Z axes).

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\,\theta & -sin\,\theta & 0 \\ 0 & sin\,\theta & cos\,\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} cos\,\theta & 0 & sin\,\theta & 0 \\ 0 & 1 & 0 & 0 \\ -sin\,\theta & 0 & cos\,\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} cos\,\theta & -sin\,\theta & 0 & 0 \\ sin\,\theta & cos\,\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



3D Rotation – Homogeneous Coordinates

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

```
void RotateX3D(float angle)
{
  Matrix4x4 rxmat = new Matrix4x4();
  rxmat.SetRow(0,new Vector4(1f,0f,              0f,              0f));
  rxmat.SetRow(1,new Vector4(0f,Mathf.Cos(angle),-Mathf.Sin(angle),0f));
  rxmat.SetRow(2,new Vector4(0f,Mathf.Sin(angle),Mathf.Cos(angle), 0f));
  rxmat.SetRow(3,new Vector4(0f,0f,              0f,              1f));
  for (int i = 0; i < vertices.Length; i++){
    vertices[i] = rxmat.MultiplyPoint(vertices[i]);
    normals[i] = rxmat.MultiplyPoint(normals[i]);
  }
  mesh.vertices = vertices;
  mesh.normals = normals;
}
```

3D Rotation – Homogeneous Coordinates

$$R_y(\theta) = \begin{bmatrix} cos\,\theta & 0 & sin\,\theta & 0 \\ 0 & 1 & 0 & 0 \\ -sin\,\theta & 0 & cos\,\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

```
void RotateY3D(float angle)
{
  Matrix4x4 rymat = new Matrix4x4();
  rymat.SetRow(0,new Vector4(Mathf.Cos(angle), 0f,Mathf.Sin(angle),0f));
  rymat.SetRow(1,new Vector4(0f,               1f,0f,              0f));
  rymat.SetRow(2,new Vector4(-Mathf.Sin(angle),0f,Mathf.Cos(angle),0f));
  rymat.SetRow(3,new Vector4(0f,               0f,0f,              1f));
  for (int i = 0; i < vertices.Length; i++){
    vertices[i] = rymat.MultiplyPoint(vertices[i]);
    normals[i] = rymat.MultiplyPoint(normals[i]);
  }
  mesh.vertices = vertices;
  mesh.normals = normals;
}
```

3D Rotation – Homogeneous Coordinates

$$R_z(\theta) = \begin{bmatrix} cos\,\theta & -sin\,\theta & 0 & 0 \\ sin\,\theta & cos\,\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

```
void RotateZ3D(float angle)
{
  Matrix4x4 rzmat = new Matrix4x4();
  rzmat.SetRow(0,new Vector4(Mathf.Cos(angle),-Mathf.Sin(angle),0f,0f));
  rzmat.SetRow(1,new Vector4(Mathf.Sin(angle), Mathf.Cos(angle),0f,0f));
  rzmat.SetRow(2,new Vector4(0f,               0f,               1f,0f));
  rzmat.SetRow(3,new Vector4(0f,               0f,               0f,1f));
  for (int i = 0; i < vertices.Length; i++){
    vertices[i] = rzmat.MultiplyPoint(vertices[i]);
    normals[i] = rzmat.MultiplyPoint(normals[i]);
  }
  mesh.vertices = vertices;
  mesh.normals = normals;
}
```

3D Rotation – Homogeneous Coordinates

- When we rotate a 3D object, we also need to apply the same rotation to the normal vectors of the object:

```
...
for (int i = 0; i < vertices.Length; i++){
  vertices[i] = rxmat.MultiplyPoint(vertices[i]);
  normals[i] = rxmat.MultiplyPoint(normals[i]);
}
mesh.vertices = vertices;
mesh.normals = normals;
...
```

- We can visualize the normals in the Unity editor:

```
private void OnDrawGizmos(){
  if (vertices == null) return;
  for (int i = 0; i < vertices.Length; i++){
    Gizmos.color = Color.yellow;
    Gizmos.DrawRay(vertices[i], normals[i]);
  }
}
```

Exercise 3

3) Rotations in 3D are performed around the origin of the coordinates system (0, 0, 0). How can we rotate a 3D object around a specific position?

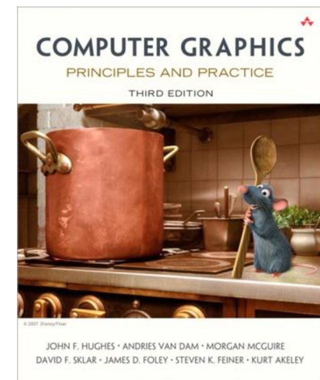
Implement in Unity the functions to rotate objects around specific positions (such as the center of the object) in the X, Y, and Z axes.

Homogeneous Coordinates

- Homogeneous coordinates are used nearly universally to represent transformations in graphics systems.
  - They underlie the design and operation of renderers implemented in all graphics hardware.
- Homogeneous coordinates not only clean up the code for transformations, but they also simplify the composition of complex transformations.

Further Reading

- Hughes, J. F., et al. (2013). **Computer Graphics: Principles and Practice** (3rd ed.). Upper Saddle River, NJ: Addison-Wesley Professional. ISBN: 978-0-321-39952-6.
  - **Chapter 10: Transformations in Two Dimensions**
  - **Chapter 11: Transformations in Three Dimensions**



- Marschner, S., et al. (2015). **Fundamentals of Computer Graphics** (4th ed.). A K Peters/CRC Press. ISBN: 978-1482229394.
  - **Chapter 6: Transformation Matrices**